Effects of compression on the vibrational modes of marginally jammed solids

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Glasses have a large excess of low-frequency vibrational modes in comparison with most crystalline solids. We show that such a feature is a necessary consequence of the weak connectivity of the solid, and that the frequency of modes in excess is very sensitive to the pressure. We analyze, in particular, two systems whose density $D(\omega)$ of vibrational modes of angular frequency ω display scaling behaviors with the packing fraction: (i) simulations of jammed packings of particles interacting through finite-range, purely repulsive potentials, comprised of weakly compressed spheres at zero temperature and (ii) a system with the same network of contacts, but where the force between any particles in contact (and therefore the total pressure) is set to zero. We account in the two cases for the observed (a) convergence of $D(\omega)$ toward a nonzero constant as $\omega \to 0$, (b) appearance of a low-frequency cutoff ω^*, and (c) power-law increase of ω^* with compression. Differences between these two systems occur at a lower frequency. The density of states of the modified system displays an abrupt plateau that appears at ω^*, below which we expect the system to behave as a normal, continuous, elastic body. In the unmodified system, the pressure lowers the frequency of the modes in excess. The requirement of stability despite the destabilizing effect of pressure yields a lower bound on the number of extra contact per particle δz: $\delta z \geqslant p^{1/2}$, which generalizes the Maxwell criterion for rigidity when pressure is present. This scaling behavior is observed in the simulations. We finally discuss how the cooling procedure can affect the microscopic structure and the density of normal modes.

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I. INTRODUCTION

The responses to thermal or mechanical excitations and the transport properties of a solid depend on its low-frequency vibrations. In a continuous elastic body, translational invariance requires that the vibrational modes, or normal modes, are acoustic modes. This leads, notably, to the Debye behavior of the density of vibrational modes $D(\omega)$: in three dimensions, $D(\omega) \sim \omega^2$ [1]. In amorphous solids structural disorder is present and this description breaks down at intermediate frequencies and length scales. In glasses, there is a large excess of low-frequency vibrations observed in neutron scattering experiments. This excess corresponds to a broad maximum in $D(\omega)/\omega^2$, called the "boson peak," which appears at frequencies on the order of a terahertz [2], that is typically between one-hundredth and one-tenth of the Debye frequency. Above the boson peak frequency, transport properties [3] are strongly affected, as in silica where the thermal conductivity is several orders of magnitude smaller than in a crystal of identical composition [4]. Furthermore, amorphous solids also exhibit force inhomogeneities that are common to granular materials [5] and glasses [6]. Thus, the resulting force propagation due to a localized perturbation differs considerably from the predictions of the continuum elasticity theory at short length scales. Recent simulations reported that this difference appears below a characteristic length scale which is directly related to the frequency of the boson peak [6]. Thus the excess modes play an important role in several anomalous properties of amorphous solids. It was recently proposed that they also govern the physics at the liquid-glass transition [7].

Despite their ubiquity, there is no accepted explanation of the underlying cause of these excess vibrational modes in glasses. A dramatic illustration of this excess was found in recent computer simulations of soft spheres with repulsive, finite-range potentials near the jamming transition [8]. At this transition an amorphous solid loses both its bulk and shear moduli and becomes a liquid [9]. In a recent paper [10] we showed how to calculate the density of states for weakly connected amorphous solids, such as those near jamming, and showed that an excess density of vibrational states is a necessary feature of such systems. In this paper we use the method of [10] to predict further consequences of compression. In particular, we explain why the coordination z must increase in a nonanalytic way with applied pressure, as observed in the simulations [11,8,5].

Because there are no attractive forces and the temperature is zero in the soft-sphere simulations of O'Hern and coworkers [5,8,12], the pressure $p=0$ at the packing fraction at the jamming transition, ϕ_c. In three dimensions for monodisperse spheres, $\phi_c \approx 0.64$. The average number of contacting neighbors per particle is z, and the elastic moduli scale has functions of p. These simulations also reveal unexpected features in the density of vibrational modes, $D(\omega)$: (a) As shown in Fig. 1, when the system is at the limit of marginal stability, as $p \to 0$, $D(\omega)$ has a plateau extending down to zero frequency with no sign of the standard ω^2 density of states normally expected for a three-dimensional solid. (b) The plateau is progressively eroded at frequencies below a characteristic frequency ω^*, that increases with the pressure p (see

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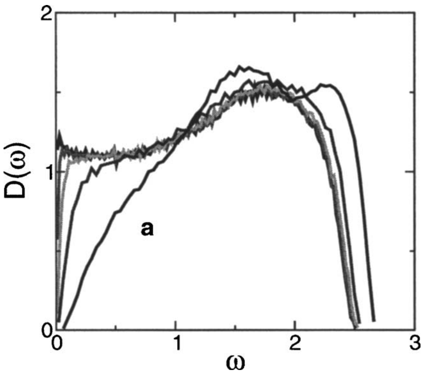

FIG. 1. The density of vibrational states, $D(\omega)$, vs angular frequency ω for the simulation of Ref. [8]. 1024 spheres interacting with repulsive harmonic potentials were compressed in a periodic cubic box to packing fraction ϕ, slightly above the jamming threshold ϕ_c. Then the energy for arbitrary small displacements was calculated and the dynamical matrix inferred. The curve labeled a is at a relative packing fraction $\phi-\phi_c=0.1$. Proceeding to the left the curves have relative volume fractions $10^{-2},10^{-3},10^{-4},10^{-8}$, respectively.

Fig. 4). (c) The value of $D(\omega)$ of the plateau is unaffected by this compression. (d) At frequencies much lower than ω^*, $D(\omega)$ still increases much faster with ω than the quadratic Debye dependence. Similar behavior was also recently seen in models of tetrahedral covalent glasses [13]. An earlier simulation of a Lennard-Jones glass had indicated that $D(\omega)$ increases at low frequencies when the coordination number z is lowered [14]. In the following we aim to relate the density of vibrational modes of weakly connected amorphous solids such as an assembly of finite-range, repulsive particles to their microscopic structure. We start by briefly reviewing the results of [10] where the scaling properties of the density of states were derived. Then we use these results to compute the effect of applied stress. We show that it leads to a nontrivial constraint on the geometry of the contact network near the jamming threshold. The scaling behavior of the soft spheres near jamming furnish a stringent test to our predictions.

At the center of our argument lies the concept of *soft modes*, or *floppy modes*. These are collective modes that conserve the distance, at first order, between any particles in contact. They have been discussed in relation to various weakly connected networks such as covalent glasses [15,16], Alexander's models of soft solids [17], models of static forces in granular packs [18,19], and rigidity percolation models, see, e.g., [20]. As we shall discuss below, they are present when a system has too low a coordination number. As a consequence, as Maxwell has shown [21], a system with a low average coordination number z has some soft modes and is therefore not rigid. There is a threshold value z_c where a system can become stable, such a state is called *isostatic*. As we shall discuss, this is the case at the jamming transition, if rattlers (particles with no contacts) are excluded. At this point there are no zero-frequency modes except for the trivial translation modes of the system as a whole. However, if any contacts were to be removed, the frequency of one mode would go to zero, that is, one soft mode would appear. As we argued in [10], this idea can be used to show that isostatic states have a constant density of states in any dimensions. When $z>z_c$, the system still behaves as an isostatic medium at a short enough length scale, which leads to the persistence of a plateau in the density of states at intermediate frequency.

The second concept we use is at the heart of Alexander's discussion of soft solids [16]. In continuum elasticity the expansion of the energy for small displacements contains a term proportional to the applied stress. It is responsible for the vibrations of strings and drumheads and also for inelastic instabilities such as the buckling of thin rods. Alexander pointed out that this term also has a strong effect at a microscopic level in weakly connected solids. For example, it confers rigidity to gels, even though these do not satisfy the Maxwell criterion for rigidity. We will show that while this term does not greatly affect the acoustic modes, it nevertheless strongly affects the soft modes. In a repulsive system of spherical particles it lowers their frequency. We argue that this can dramatically change the density of states at low frequency, as confirmed by a comparison of simulations where the force in any contact is present, or set to zero. We show that these considerations also lead to a relation between the excess connectivity $\delta z \equiv z-z_c$ and the pressure p.

The paper is organized as follows. In the next section, we write the expansion of the elastic energy that we use to derive the soft-modes equation. We then define and discuss the isostatic case, and the nature of the soft modes that appear when contacts are removed in such a system. In the third part we compute the density of states when the effect of the applied stress on the vibrations is neglected. This approximation corresponds to a real physical system: a network of relaxed springs. We use the soft modes in a variational argument to show that an isostatic state has a constant density of vibrational modes. We extend this argument to include the case where the coordination number increases above z_c, as is the case of the soft-sphere system under isotropic compression. We show that such a system behaves as an isostatic one for length scales smaller than $l^* \sim \delta z^{-1}$. This leads to a plateau in the density of states for frequency higher than $\omega^* \sim \delta z$. At lower frequencies we expect the system to behave as a continuous medium with a Debye regime, which is consistent with our simulations. In the fourth part we study the effect of an applied pressure on $D(\omega)$. We show that although it does not affect the acoustic modes, it lowers the frequency of the soft modes. We give a simple scaling argument to evaluate this effect, and discuss its implication for the density of states. Incidentally this also furnishes an inequality between δ_z and the pressure, which is also verified by the simulations, and which generalizes the Maxwell criterion for rigidity. Finally we discuss the influence of the cooling rate and the temperature history on the spatial structure and the density of states of the system, and conclude our work.

II. SOFT MODES AND ISOSTATICITY

A. Energy expansion

Following [8] we consider N soft spheres packed into a spatially periodic cubic cell of side L at volume fraction ϕ.

To describe the vibrations of a system of particles, we expand the energy around equilibrium. For a central interaction $V(r)$ one obtains

$$\delta E = \sum_{ij} V'(r_{ij}^{eq}) dr_{ij} + \tfrac{1}{2} V''(r_{ij}^{eq}) dr_{ij}^2 + O(r_{ij}^3), \quad (1)$$

where the sum is over all pairs of particles and r_{ij}^{eq} is the equilibrium distance between particles i and j. In order to get an expansion in terms of the particle displacements from equilibrium $\delta\vec{R}_1, \ldots, \delta\vec{R}_N$ we use

$$dr_{ij} = (\delta\vec{R}_j - \delta\vec{R}_i) \cdot \vec{n}_{ij} + \frac{[(\delta\vec{R}_j - \delta\vec{R}_i)^{\perp}]^2}{2r_{ij}^{eq}} + O(\delta\vec{R}^3), \quad (2)$$

where $\vec{n}_{ij}$ is the unit vector along the direction ij and $(\delta\vec{R}_j - \delta\vec{R}_i)^{\perp}$ indicates the projection of $\delta\vec{R}_j - \delta\vec{R}_i$ on the plane orthogonal to $\vec{n}_{ij}$. When Eq. (2) is used in Eq. (1), the linear term in the displacement field disappears (the system is at equilibrium) and we obtain

$$\delta E = \sum_{ij} V'(r_{ij}^{eq}) \frac{[(\delta\vec{R}_j - \delta\vec{R}_i)^{\perp}]^2}{2r_{ij}^{eq}} \quad (3)$$

$$+ \tfrac{1}{2} V''(r_{ij}^{eq}) [(\delta\vec{R}_j - \delta\vec{R}_i) \cdot \vec{n}_{ij}]^2 + O(\delta\vec{R}^3). \quad (4)$$

In what follows we consider repulsive, finite-range "soft spheres." For the interparticle distance $r < \sigma$, the particles have nonzero mutual energy and are said to be in contact. They interact with the following potential:

$$V(r) = \frac{\epsilon}{\alpha}\left(1 - \frac{r}{\sigma}\right)^{\alpha}, \quad (5)$$

where σ is the particle diameter and ϵ a characteristic energy. For $r > \sigma$ the potential vanishes and particles do not interact. Henceforth we express all distances in units of σ, all energies in units of ϵ, and all masses in units of the particle mass m. In the following, we consider the harmonic case $\alpha = 2$. In Sec. V we argue that these results can be extended, for example, to the case of Hertzian contacts [8] where $\alpha = \frac{5}{2}$. In the harmonic case we have

$$\delta E = \left(\frac{1}{2} \sum_{\langle ij \rangle} (r_{ij}^{eq} - 1) \frac{[(\delta\vec{R}_j - \delta\vec{R}_i)^{\perp}]^2}{2r_{ij}^{eq}} \right) \quad (6)$$

$$+ \frac{1}{2} \sum_{\langle ij \rangle} [(\delta\vec{R}_j - \delta\vec{R}_i) \cdot \vec{n}_{ij}]^2 + O(\delta\vec{R}^3), \quad (7)$$

where the sum is over all N_c contacts $\langle ij \rangle$. It is convenient to express Eq. (6) in matrix form, by defining the set of displacements $\delta\vec{R}_1, \ldots, \delta\vec{R}_N$ as a dN-component vector $|\delta\mathbf{R}\rangle$. Then Eq. (6) can be written in the form $\delta E = \langle \delta\mathbf{R} | \mathcal{M} | \delta\mathbf{R} \rangle$. The corresponding matrix $\mathcal{M}$ is known as the dynamical matrix [1]. The dN eigenvectors of the dynamical matrix are the normal modes of the particle system, and its eigenvalues are the squared angular frequencies of these modes.

The first term in Eq. (6) is proportional to the contact forces. In the rest of this paper we shall refer to this term as the *stress term* or *transverse term*. Near the jamming transition $r_{ij}^{eq} \to 1$ so that this term becomes arbitrarily small. We start by neglecting it, and we come back to its effects in the last section. This approximation corresponds to a real physical system where the soft spheres are replaced by point particles interacting with relaxed springs. We now have

$$\delta E = \frac{1}{2} \sum_{\langle ij \rangle} [(\delta\vec{R}_j - \delta\vec{R}_i) \cdot \vec{n}_{ij}]^2, \quad (8)$$

where $\mathcal{M}$ can be written as an N by the N matrix whose elements are themselves tensors of rank d, the spatial dimension:

$$\mathcal{M}_{ij} = -\delta_{\langle ij \rangle} \vec{n}_{ij} \otimes \vec{n}_{ij} + \delta_{i,j} \sum_{\langle l \rangle} \vec{n}_{il} \otimes \vec{n}_{il}, \quad (9)$$

where $\delta_{\langle ij \rangle} = 1$ when i and j are in contact and the sum is taken on all the contacts l with i.

B. Soft modes

If the system has too few contacts, $\mathcal{M}$ has a set of modes of vanishing restoring force and thus vanishing vibrational frequency. These are the *soft modes* mentioned in the Introduction. For these soft modes the energy δE of Eq. (8) must vanish; therefore they must satisfy the N_c constraint equations:

$$(\delta\vec{R}_i - \delta\vec{R}_j) \,.\, \vec{n}_{ij} = 0 \text{ for all } N_c \text{ contacts } \langle ij \rangle. \quad (10)$$

This linear equation defines the vector space of displacement fields that conserve the distances at first order between particles in contact. The particles can yield without restoring force if their displacements lie in this vector space. Equation (10) is purely geometrical and does not depend on the interaction potential. Each equation restricts the dN-dimensional space of $|\delta\mathbf{R}\rangle$ by one dimension. In general, these dimensions are independent, so that the number of independent soft modes is $dN - N_c$. Of these, $d(d+1)/2$ modes are dictated by the translational and rotational invariance of the energy function $\delta E[\delta\mathbf{R}]$. Apart from these, there are $dN - N_c - d(d+1)/2$ independent internal soft modes.

C. Isostaticity

There are no internal soft modes in a rigid solid. This is true when a system of repulsive spheres jams, when the rattlers (i.e., particles without contacts) are removed. Therefore jammed states must satisfy $N_c \geq dN - d(d+1)/2$, which is the Maxwell criterion for rigidity. In fact at the jamming transition, if rattlers [22] are removed, one can show this inequality becomes an equality [18,19,23], as was verified in [5]. Such a system is called *isostatic*. The coordination number z is then $z_c \equiv 2N_c/N \to 2d$.

An isostatic system is marginally stable: if q contacts are cut, a space of soft modes of dimension q appears. For our coming argument we need to discuss the extended character of these modes. In general, when only one contact $\langle ij \rangle$ is cut in an isostatic system, the corresponding soft mode is not localized near $\langle ij \rangle$. This arises from the nonlocality of the

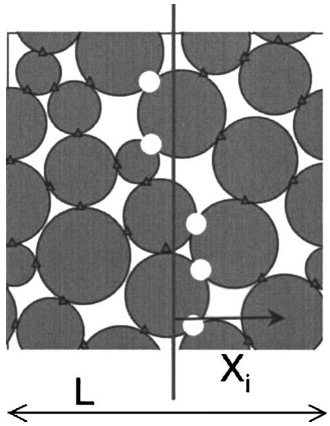

FIG. 2. Illustration of the boundary contact removal process described in the text. 18 particles are confined in a square box of side L periodically which is continued horizontally and vertically. An isostatic packing requires 33 contacts in this two-dimensional system. An arbitrarily drawn vertical line divides the system. A contact is removed wherever the line separates the contact from the center of a particle. 28 small triangles mark the intact contacts; removed contacts are shown by the five white circles.

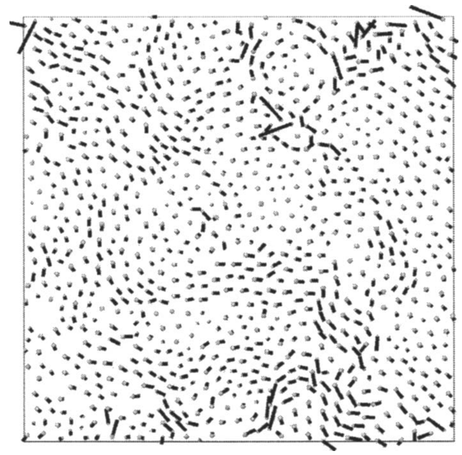

FIG. 3. One soft mode in two dimensions for $N \approx 1000$ particles. Each particle is represented by a dot. The relative displacement of the soft mode is represented by a line segment extending from the dot. The mode was created from a previously prepared isostatic configuration, periodic in both directions, following [8]. 20 contacts along the vertical edges were then removed and the soft modes determined. The mode pictured is an arbitrary linear combination of these modes.

isostatic condition that gives rise to the soft modes; as was confirmed in the isostatic simulations of Ref. [18], which observed that the amplitudes of the soft modes were spread over a nonzero fraction of the particles. When many contacts are severed, the extended character of the soft modes that appear depends on the geometry of the region being cut. If this region is compact many of the soft modes are localized. For example, cutting all the contacts inside a sphere totally disconnects each particle within the sphere. Most of the soft modes are then the individual translations of these particles and are not extended throughout the system.

In what follows we will be particularly interested in the case where the region of the cut is a hyperplane as illustrated in Fig. 2. In this situation occasionally particles in the vicinity of the hyperplane can be left with less than d contacts, so that trivial localized soft modes can also appear. However we expect that there is a nonvanishing fraction q' of the total soft modes that are not localized near the hyperplane, but rather extend over the entire system, like the mode shown in Fig. 3. We shall define extended modes more precisely in the next section.

III. $D(\omega)$ IN A SYSTEM OF RELAXED SPRINGS

A. Isostatic case

1. Variational procedure

We aim to show first that the density of states of an isostatic system does not vanish at zero frequency. $D(\omega)$ is the total number of modes per unit volume per unit frequency range. Therefore we have to show that there are at least on the order of ωL^d normal modes with frequencies smaller than ω for any small ω in a system of linear size L. As we justify later, if proven in a system of size L for $\omega \sim \omega_L \sim 1/L$, this property can be extended to a larger range of ω independent of L. Therefore, it is sufficient to show that they are of the order of L^{d-1} normal modes with frequency of the order of $1/L$, instead of the order of one such mode in a continuous solid.

To do so we use a variational argument: $\mathcal{M}$ is a positive symmetric matrix. Therefore if a normalized mode has an energy δE, we know that the lowest eigenmode has a frequency $\omega_0 \leqslant \sqrt{\delta E}$. Such an argument can be extended to a set of modes [24]: if there are m *orthonormal* trial modes with energy $\delta E \leqslant \omega_t^2$, then there are at least $m/2$ eigenmodes with frequencies smaller than $\sqrt{2}\omega_t$. Therefore, we are led to find to the order of L^{d-1} trial orthonormal modes with energy of order $1/L^2$.

2. Trial modes

Our procedure for identifying the lowest frequency modes resembles that used for an ordinary solid. An isolated block of solid has three soft modes that are simply translations along the three coordinate axes. If the block is enclosed in a rigid container, translation is no longer a soft mode. However, one may recover the lowest-frequency, fundamental modes by making a smooth, sinusoidal distortion of the original soft modes. We follow an analogous procedure to find the fundamental modes of our isostatic system. First we identify the soft modes associated with the boundary constraints by removing these constraints. Next we find a smooth, sinusoidal distortion of these modes that allows us to restore these constraints.

For concreteness we consider the three-dimensional cubical N-particle system $\mathcal{S}$ of Ref. [8] with periodic boundary conditions at the jamming threshold. We label the axes of the cube by x, y, and z. $\mathcal{S}$ is isostatic, so that the removal of n contacts allows exactly n displacement modes with no restoring force. Consider, for example, the system $\mathcal{S}'$ built from $\mathcal{S}$ by removing the $q \sim L^2$ contacts crossing an arbitrary plane orthogonal to (ox); by convention at $x=0$, see Fig. 2. $\mathcal{S}'$, which has a free boundary condition instead of periodic ones along (ox), contains a space of soft modes of dimension q [25], instead of one such mode—the translation of the whole system—in a normal solid. As stated above, we suppose that a subspace of dimension $q' \sim L^2$ of these soft modes contains only extended modes. We define the *extension* of a mode relative to the cut hyperplane in terms of the amplitudes of the mode at distance x from this hyperplane. Specifically the extension e of a normalized mode $|\delta\mathbf{R}\rangle$ is defined by $\Sigma_i \sin^2(x_i\pi/L)\langle i|\delta\mathbf{R}\rangle^2 = e$, where the notation $\langle i|\delta\mathbf{R}\rangle$ indicates the displacement of the particle i of the mode considered. For example, a uniform mode with $\langle i|\delta\mathbf{R}\rangle$ constant for all sites has $e=\frac{1}{2}$ independent of L. On the other hand, if $\langle i|\delta\mathbf{R}\rangle=0$ except for a site i adjacent to the cut hyperplane, $x_i/L \sim L^{-1}$ and $e \sim L^{-2}$. We define the subspace of extended modes by setting a fixed threshold of extension e_0 of order 1 and thus including only soft modes β for which $e_\beta > e_0$. As we discussed in the last section, we expect that a fixed fraction of the soft modes remain extended as the system becomes large. Thus if q' is the dimension of the extended modes vector space, we shall suppose that q'/q remains finite as $L \to \infty$. The Appendix presents our numerical evidence for this behavior.

We now use the vector space of dimension $q' \sim L^2$ of extended soft modes of $\mathcal{S}'$ to build q' orthonormal trial modes of $\mathcal{S}$ of frequency of the order $1/L$. Let us define $|\delta\mathbf{R}_\beta\rangle$ to be a normalized basis of this space, $1 \leq \beta \leq q'$. These modes are not soft in the jammed system $\mathcal{S}$ since they deform the previous q contacts located near $x=0$. Nevertheless, a set of trial modes $|\delta\mathbf{R}^*_\beta\rangle$ can still be formed by altering the soft modes so that they do not have an appreciable amplitude at the boundary where the contacts were severed. We seek to alter the soft mode to minimize the distortion at the severed contacts while minimizing the distortion elsewhere. Accordingly, for each soft mode β we define the corresponding trial-mode displacement $\langle i|\delta\mathbf{R}^*\rangle$ to be

$$\langle i|\delta\mathbf{R}^*_\beta\rangle \equiv C_\beta \sin\left(\frac{x_i\pi}{L}\right)\langle i|\delta\mathbf{R}_\beta\rangle, \tag{11}$$

where the constants C_β are introduced to normalize the modes. C_β depends on the spatial distribution of mode β. If, for example, a highly localized mode has $\langle i|\delta\mathbf{R}\rangle=0$ except for a site i adjacent to the cut plane, C_β grows without bound as $L \to \infty$. In the case of extended modes $C_\beta^{-2} \equiv \Sigma_{\langle ij\rangle}\sin^2(x_i\pi/L)\langle j|\delta\mathbf{R}_\beta\rangle^2 = e_\beta > e_0$, and therefore C_β is bounded above by $e_0^{-1/2}$. The sine factor suppresses the problematic gaps and overlaps at the q contacts near $x=0$ and $x=L$. The unit basis $|\delta\mathbf{R}_\beta\rangle$ can always be chosen such that the $|\delta\mathbf{R}^*_\beta\rangle$ are orthogonal, simply because the modulation by a sine that relates the two sets is an invertible linear mapping in the subspace of extended modes. Furthermore, one readily verifies that the energy of each $|\delta\mathbf{R}^*_\beta\rangle$ is small, and that the sine modulation generates an energy of order $1/L^2$ as expected. Indeed we have from Eq. (8)

$$\delta E = C_\beta^2 \sum_{\langle ij\rangle}\left[\left(\sin\left(\frac{x_i\pi}{L}\right)\langle i|\delta\mathbf{R}_\beta\rangle - \sin\left(\frac{x_j\pi}{L}\right)\langle j|\delta\mathbf{R}_\beta\rangle\right)\cdot\vec{n}_{ij}\right]^2. \tag{12}$$

Using Eq. (10), and expanding the sine, one obtains

$$\delta E \approx C_\beta^2 \sum_{\langle ij\rangle}\cos^2\left(\frac{x_i\pi}{L}\right)\frac{\pi^2}{L^2}(\vec{n}_{ij}\cdot\vec{e}_x)^2(\langle j|\delta\mathbf{R}_\beta\rangle\cdot\vec{n}_{ij})^2 \tag{13}$$

$$\leq e_0^{-1}(\pi/L)^2\sum_{\langle ij\rangle}\langle j|\delta\mathbf{R}_\beta\rangle^2, \tag{14}$$

where $\vec{e}_x$ is the unit vector along (ox), and where we used $|\cos| \leq 1$. The sum on the contacts can be written as a sum on all the particles since only one index is present in each term. Using the normalization of the mode β and the fact that the coordination number of a sphere is bounded by a constant z_{max} ($z_{max}=12$ for three-dimensional spheres [26]), one obtains

$$\delta E \leq e_0^{-1}(\pi/L)^2 z_{max} \equiv \omega_L^2. \tag{15}$$

We have found on the order of L^2 the trial orthonormal modes of frequency bounded by $\omega_L \sim 1/L$, and we can apply the variational argument mentioned above: the average density of states is bounded below by a constant below frequencies of the order ω_L. In what follows, the trial modes introduced in Eq. (11), which are the soft modes modulated by a sine wave, shall be called "anomalous modes."

To conclude, one may ask if this variational argument can be improved, for example, by considering geometries of broken contacts different from the hyperplane surfaces we have considered so far. When contacts are cut to create a vector space of extended soft modes, the soft modes must be modulated with a function that vanishes where the contacts are broken in order to obtain trial modes of low energy. On the one hand, cutting many contacts increases the number of trial modes. On the other hand, if too many contacts are broken, the modulating function must have many "nodes" where it vanishes. Consequently this function displays larger gradients and the energies of the trial modes increase. Cutting a surface (or many surfaces, as we shall discuss below) is the best compromise between these two opposite effects. Thus our argument gives a natural limit to the number of low-frequency states to be expected.

3. Extension to a wider range of frequencies

We may extend this argument to show that the bound on the average density of states extends to higher frequencies. If the cubic simulation box were now divided into m^3 subcubes of size L/m, each subcube must have a density of states equal to the same $D(\omega)$ as was derived above, but extending to frequencies on order of $m\omega_L$. These subsystem modes must be present in the full system as well, therefore the bound on $D(\omega)$ extends to $[0, m\omega_L]$. We thus prove that the same bound on the average density of states holds down to sizes of the order of a few particles, corresponding to frequencies independent of L. We note that in d dimensions this argument may be repeated to yield a total number of modes

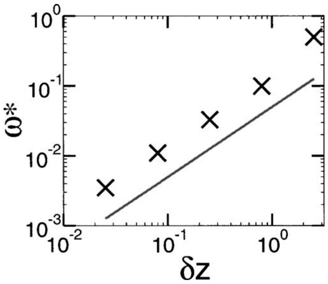

FIG. 4. Scaling of ω^* with the excess coordination number δz in the system with relaxed springs. The line has a slope one.

L^{d-1} below a frequency $\omega_L \approx 1/L$, thus yielding a limiting nonzero density of states in any dimension.

We note that the trial modes of energy $\delta E \sim l^{-1}$ that we introduce by cutting the full system into subsystems of size l are, by construction, localized to a distance scale l. Nevertheless we expect that these trial modes will hybridize with the trial modes of other, neighboring, subsystems; the corresponding normal modes will, therefore, not be localized to such short length scales.

B. State with $\delta z > 0$

When the system is compressed and moves away from the jamming transition, the simulations show that the extra coordination number $\delta z \equiv z - z_c$ increases. In the simulation, the compression also increases the force at all contacts. However, in this section we will ignore these forces and focus our attention only on the contact network created by the compression. Any tension or compression in the contacts is removed. The effect on the energy is to remove the first bracketed term in Eq. (6) above. We note that removing these forces, which must add to zero on each particle, does not disturb the equilibrium of the particles or create displacements. In this section we ignore the question of how δz depends on the degree of compression. We will return to this question in the next section.

Compression causes $\Delta N_c = N\delta z/2 \sim L^d \delta z$ extra constraints to appear in Eq. (8). Cutting the boundaries of the system, as we did above, relaxes $q \sim L^{d-1}$ constraints. For a large system, $L^d \delta z > L^{d-1}$ and thus $q < \Delta N_c$. Thus Eq. (8) is still overconstrained and there will be no soft modes in the system. However, as the systems become smaller, the excess number of constraints, ΔN_c, diminishes; for L smaller than some $l^* \sim \delta z^{-1}$, q becomes larger than ΔN_c the system is again under-constrained as was already noticed in [18]. This allows one to build low-frequency modes in subsystems smaller than l^*. These modes appear above a cut-off frequency $\omega^* \sim l^{*-1}$; they are the "anomalous modes" that contribute to the flat plateau in $D(\omega)$ above ω^*. In other words, anomalous modes with characteristic length smaller than l^* are not affected very much by the extra contacts, and the density of states is unperturbed above a frequency $\omega^* \sim \delta z$. This scaling is checked numerically in Fig. 4. This prediction is in very good agreement with the data up to $\delta z \approx 2$.

At frequencies lower than ω^* we expect the system to behave as a disordered, but not poorly connected, elastic medium. We expect that the vibrational modes will be similar to the plane waves of a continuous elastic body. We refer to these modes as "acoustic modes." Thus we expect $D(\omega)$ at small ω to vary as $\omega^{d-1}c^{-d}$, where $c(\delta z)$ is the sound speed at the given compression. This c may be inferred from the bulk and shear moduli measured in the simulations [15,8,11]; one finds the transverse velocity $c_t \sim (\delta z)^{1/2}$, and the longitudinal velocity $c_l \sim \delta z^0$ in both three and two dimensions. Thus at low frequency $D(\omega)$ is dominated by the transverse acoustic modes and at $\omega = \omega^*$ the acoustic density of states is $\omega^{d-1}c_t^{-d} \sim \delta z^{d-1}\delta z^{-d/2} \sim \delta z^{d/2-1}$. For a three-dimensional system the acoustic density of states should be dramatically smaller than the plateau density of states. Since there is no smooth connection between the two regimes we expect a sharp dropoff in $D(\omega)$ for $\omega < \omega^*$. Such a dropoff is indeed observed, as seen for a three-dimensional system in Fig. 5. In fact, because of the finite size of the simulation, no acoustic modes are apparent at $\omega < \omega^*$ near the transition.

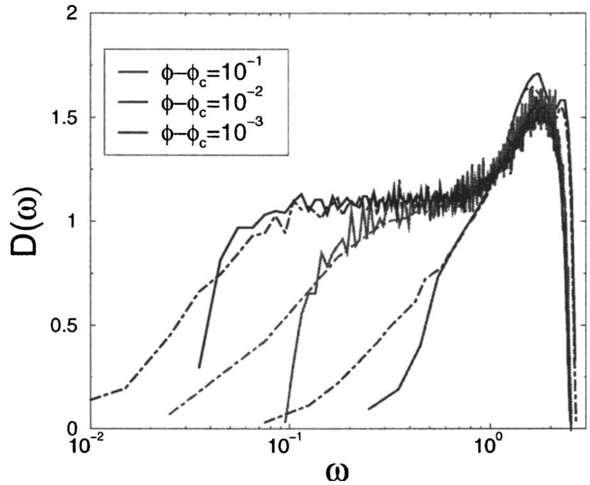

FIG. 5. Log-linear plot of the density of states for a three-dimensional system with $N=1024$ for three values of $\phi-\phi_c$ in the soft-sphere system (dotted line) and the system where the applied stress term has been removed (solid line).

The behavior of such systems near the jamming threshold thus depends on the frequency ω at which they are observed. For $\omega > \omega^*$ the system behaves as an isostatic system, and for $\omega < \omega^*$ it behaves as a continuous elastic medium. Since the transverse and the longitudinal velocities do not scale in the same way, the presence of a unique crossover in frequency leads to the appearance of two distinct length scales l_l and l_t, defined as $l_l \sim c_l \omega^{*-1}$ and $l_t \sim c_t \omega^{*-1}$. These lengths correspond, respectively, to the wavelengths of the longitudinal and transverse acoustic modes at ω^*. Note that since $c_l \sim \delta z^0$, one has $l_l \sim l^*$. Interestingly, $l_t \sim \delta z^{-1/2}$ is the smallest system size for which acoustic modes exist. For smaller system sizes, the lowest frequency mode is not a plane wave, but is an anomalous mode. l_t can be observed numerically by considering the peak of the transverse structure factor at ω^* [12].

Our argument ignores the spatial fluctuations of δz. If these fluctuations were spatially uncorrelated they would be Gaussian upon coarse graining: then the extra number of contacts ΔN_c in a subregion of size L would have fluctuations of order $L^{d/2}$. The scaling of the contact number that appears in our description is $\Delta N_c \sim L^{d-1}$ and is therefore larger than these Gaussian fluctuations for $d>2$. In other terms at the length scale l^*, where soft modes appear, the fluctuations in the number of contacts inside the bulk are negligible in comparison with the number of contacts at the surface. Therefore the anomalous modes are not sensitive to fluctuations in coordination numbers in three dimensions near the transition. In Sec. V we will argue that there are spatial anticorrelations in z, so that fluctuations also do not affect the extended soft modes in two dimensions.

Note that these arguments do not preclude the existence of low-frequency localized modes that may appear in regions of small size $l \ll l^*$, and that could be induced by very weak local coordination or specific arrangements of the particles. The presence of such modes would increase the density of states at low frequency. There is no evidence of their presence in the simulations of [8].

IV. EFFECTS OF THE APPLIED STRESS ON VIBRATIONAL MODES

In this section we describe how the above simple description of $D(\omega)$ is affected by the presence of an applied stress. In general, when a system of particles at equilibrium is formed, there are forces between interacting particles. For harmonic soft spheres it leads to a nonvanishing first term in Eq. (6) $\frac{1}{4}\Sigma_{\langle ij\rangle}(r_{ij}^{eq}-1)[(\delta\vec{R}_j-\delta\vec{R}_i)^{\perp}]^2$, where we used $r_{ij}\approx 1$. This term is (a) negative for repulsive particles, (b) proportional to the transverse relative displacement between particles in contact, and (c) scales linearly with the pressure p, and therefore vanishes at the jamming transition. The full dynamical matrix $\mathcal{D}$ can be written

$$\mathcal{D} = \mathcal{M} + \mathcal{M}', \tag{16}$$

where $\mathcal{M}'$ is written in tensorial notation in the footnote [27]. The spectrum of $\mathcal{D}$ has it *a priori* no simple relation with the spectrum of $\mathcal{M}$. Because $\mathcal{M}'$ is much smaller than $\mathcal{M}$ near the transition, one can successfully use perturbation theory for the bulk part of the normal modes of $\mathcal{M}$. However, perturbation theory fails at very low frequency, which is of most interest to us here. In this region the spectrum of $\mathcal{M}$ contains the acoustic modes and the anomalous modes forming the plateau. In what follows we estimate the change of frequency induced by the applied stress on these modes. We show that the relative correction to the plane-wave frequencies is very small, whereas the frequency of the anomalous modes can be changed considerably. Finally we show that these considerations lead to a correction to the Maxwell rigidity criterion.

A. Applied stress and acoustic modes

Consider a plane wave of wave vector k. As the directions $\vec{n}_{ij}$ are random, both the relative longitudinal and transverse displacements of this plane wave are of the same order: $[(\delta\vec{R}_i-\delta\vec{R}_j)^{\perp}]^2 \sim [(\delta\vec{R}_i-\delta\vec{R}_j)\cdot\vec{n}_{ij}]^2 \sim k^2$. Consequently the relative correction $\Delta E/E$ induced by the applied stress term is very small:

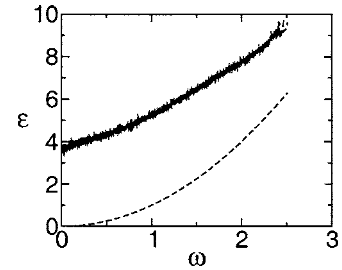

FIG. 6. Sum of the transverse terms (full curve) $\epsilon=\frac{1}{2}\Sigma_{\langle ij\rangle}[(\delta\vec{R}_j-\delta\vec{R}_i)^{\perp}]^2$ and longitudinal terms (dotted curve) $\epsilon=\frac{1}{2}\Sigma_{\langle ij\rangle}[(\delta\vec{R}_j-\delta\vec{R}_i)\cdot\vec{n}_{ij}]^2$ for each mode of frequency ω at the jamming threshold in three dimensions. The longitudinal term is equal to the energy of the modes and vanish quadratically at 0 frequency. The transverse term converges toward a constant different from 0.

$$\frac{\Delta E}{E} \approx \frac{\frac{1}{2}\sum_{\langle ij\rangle}(r_{ij}^{eq}-1)[(\delta\vec{R}_j-\delta\vec{R}_i)^{\perp}]^2}{\sum_{\langle ij\rangle}[(\delta\vec{R}_j-\delta\vec{R}_i)\cdot\vec{n}_{ij}]^2} \tag{17}$$

since $r_{ij}^{eq}-1$ is proportional to the pressure p, while the other factors remain constant as $p\to 0, \Delta E/E \sim p$, and is thus arbitrarily small near the jamming threshold [28].

B. Applied stress and anomalous modes

For anomalous modes the situation is very different: we expect the transverse relative displacements to be much larger than the longitudinal ones. Indeed soft modes were built by imposing zero longitudinal terms, but there were no constraints on the transverse ones. These are the degrees of freedom that generate the large number of soft modes. The most simple assumption is that the relative transverse displacements are of the order of the displacements themselves, that is $\Sigma_{\langle ij\rangle}[(\delta\vec{R}_j-\delta\vec{R}_i)^{\perp}]^2 \sim \Sigma_i \delta\vec{R}_i^2=1$ for the anomalous modes that appear above ω^*. This estimate can be checked numerically for an isostatic system where this sum is computed for all ω. The sum of the transverse relative displacements converges to a constant when $\omega\to 0$ as assumed, see Fig. 6.

Finally we can estimate the scaling of the correction in the energy ΔE induced by the stress term on the anomalous modes:

$$\Delta E \sim -\sum_{\langle ij\rangle}(1-r_{ij}^{eq})[(\delta\vec{R}_j-\delta\vec{R}_i)^{\perp}]^2 \sim -p \tag{18}$$

which is an *absolute* correction, and which can be nonnegligible in comparison with the energy E.

C. Onset of the anomalous modes

We can now estimate the lowest frequency of the anomalous modes. The modes that appear at ω^* in the relaxed-spring system have an energy lowered by an amount on the order of $-p$ in comparison to the original system. Applying the variational theorem of the last section to the collection of slow modes near ω^*, one finds that there must be slow normal modes with a lower energy. That is, the energy ω_{AM}^2 at which anomalous modes appear satisfies

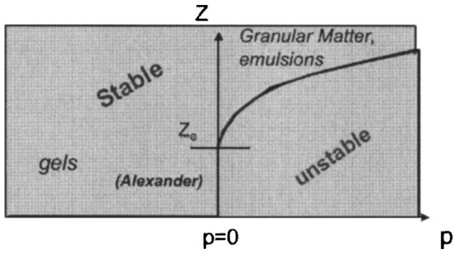

FIG. 7. Phase diagram of rigidity in terms of the coordination number and pressure. When $p>0$, the line separating the stable and unstable regions is defined by Eq. (20). When the pressure is negative, any connected system can be rigid.

$$\omega_{AM}{}^2 \leqslant \omega^{*2} - A_2 p \equiv A_1 \delta z^2 - A_2 p, \tag{19}$$

where A_1 and A_2 are two positive constants. Thus coordination and stress determine the onset of the excess modes.

D. Extended Maxwell criterion for the contact number under stress

From this estimate we can readily obtain a relation between the coordination and pressure that guarantees the stability of a system. There should be no negative frequencies in a stable system, therefore $\omega_{AM}>0$. Thus in an harmonic system the right-hand side of Eq. (19) must be positive:

$$\delta z \leqslant C_0 p^{1/2} \equiv \delta z_{min}, \tag{20}$$

where C_0 is a constant. This inequality, which must hold for any spatial dimension, indicates how a system must be connected to counterbalance the destabilizing effect of the pressure. A phase diagram of rigidity is represented in Fig. 7. When $p=0$, the minimal coordination z_c corresponds to the isostatic state: this is the Maxwell criterion. As we said, for spherical particles $z_c=2d$. When friction is present, one finds $z_c=d+1$. When $p>0$, Eq. (20) delimits the region of rigid systems: granular matter, emulsions lie above this line. As was shown by Alexander [17], when $p<0$, even systems with many fewer contacts than required by the Maxwell criterion are rigid. These systems contain many soft modes as defined in Eq. (10), but they are all stabilized by the positive bracketed term of Eq. (6). This is the case, for example, in a gel where polymers are stretched by the osmotic pressure of the solvent. Thus the network of reticulated polymers carries a negative pressure, which rigidifies the system and leads to a nonvanishing shear modulus. Note that a similar phase diagram, with the same singularity of δz but a different z_c, was obtained by a mean field approach [29].

The relation (20) is verified in the simulations of [8] where $\delta z \sim p^{1/2}$: the numerical results are in agreement with an *equality* of Eq. (20). Furthermore it appears in Fig. 5 that $\omega^* \geqslant \omega_{AM}$. In other words, the two opposite effects of pressure on the vibrations, that is (i) the increase in the coordination and (ii) the addition of a negative term in the energy expansion, compensate [30]. In the discussion section we justify why the system of [2] is thus marginally stable even above the transition when $p>0$, and we furnish examples of dynamics that lead to such features.

V. DISCUSSION

A. Stronger constraints on δz

The simulations of [2] show $\delta z \sim p^{1/2}$, thus potentially saturating the bound of Eq. (20), so that there are excess modes extending to frequencies much lower than ω^*. Here we furnish an example of dynamics that lead to such a situation. Consider an initial condition where forces are balanced on every particle, but such that the inequality (20) is not satisfied. Consequently, this system is not stable: infinitesimal fluctuations make the system relax with the collapse of unstable modes. Such dynamics were described by Alexander in [17] as structural buckling events: these events are induced by a positive stress as occurs in the buckling of a rod; but here they take place in the bulk of an amorphous solid. These events *a priori* create both new contacts and decrease the pressure. When the bound of (20) is reached, there are no more unstable modes. If the temperature is zero, the dynamics stop. Consequently one obtains a system where Eq. (20) is an *equality*. Therefore (i) this system is weakly connected and (ii) $\omega_{AM} \approx 0$, so that there are anomalous modes very different from acoustic modes extending to zero frequency. A similar argument is present in [24].

In the simulations of Ref. [8] the relaxation proceeds in the following way. The system is initially in equilibrium at a high temperature. Then it is quenched instantaneously to zero temperature. At short time scales the dynamics that follows is dominated by the relaxation of the stable, high frequency modes. The main effect is to restore approximately force balance on every particle. At this point, if the inequality (20) is satisfied, then the dynamics stop. However, if it is not satisfied then we are in the situation of buckling as described above. The pressure and coordination number continue to change until the last unstable mode has been stabilized. At this point the bound of Eq. (20) is marginally satisfied, and there is no driving force for further relaxation.

The data of [2] were obtained by gradually decreasing the pressure from this initial state of zero temperature and nonzero pressure. The reduction of pressure causes some contacts to open. The opening of these contacts tends to destabilize normal modes and reduce their frequencies, while the reduction in pressure tends to stabilize them. If the particles simply spread apart affinely, the destabilizing effect would be expected to dominate [32]. Thus we argue that some incremental buckling must occur as in the initial temperature quench. The buckling increases the contact number and decreases the pressure until marginal stability is achieved, so that the inequality of Eq. (20) is marginally satisfied as the pressure decreases.

It is interesting to discuss further which systems, and which procedures, can have marginal stable states. In repulsive short-range systems, we expect the situation of marginal stability that follows an infinite cooling rate to take place for a domain of the parameters of initial conditions (ϕ, T), located at high temperature and low density. This domain might stop at a finite ϕ even when the temperature is infinite. This is true for particles interacting with a Gaussian potential, as was shown in simulations and theoretical analysis on Euclidian random matrices [33]. There most of the unstable modes vanish at a finite ϕ despite $T=\infty$.

When the cooling rate is finite we expect that the relaxation does not stop when all the modes are stable. For example activated events can, in principle, lead to the collapse of anomalous modes. These events *a priori* increase the connectivity and decrease the pressure further than the bound of Eq. (20), leading to $\omega_{AM}>0$. Interestingly hyperquenched mineral glasses show a much larger number of excess modes [34] in comparison with normally cooled glasses, whereas annealed polymeric glasses show a diminished excess of such modes [35].

B. Fluctuation of connectivity

In Sec. III we argued that although fluctuations were negligible in three dimensions near the transition, they might have effects in two dimensions. Here we address these possible effects. In other systems such as ferromagnets the spatial fluctuations of, e.g., magnetic coupling, lead to noticeable and sometimes striking effects [36]. It is natural to ask what analogous effects might be induced by fluctuations of contact number z in the system of [8] in two dimensions. In a system with uncorrelated contacts in two dimensions, the modes in the weakly connected regions would be softer than those in the more strongly connected regions. This would be expected to create modes with frequencies below the nominal ω^* in the system of relaxed springs of Sec. III, and would imply the presence of modes below the bound ω_{AM} in the original system. We argue that in the simulations of [8] such fluctuations are negligible. It is important to note that z is not an uncorrelated random variable like the magnetic coupling mentioned above. For example, if z were an uncorrelated random variable, one would expect that the fluctuations in the total number of contacts Nz would be of order $\sqrt{N}$. However, at the jamming threshold the system is isostatic and the number of contacts is precisely $2dN$; there are no fluctuations. More generally, the bound on δz in the last section applies not merely on average but rigorously to any subregion of a size larger than l^*, since all subregions experience the same pressure p. Furthermore for a marginally stable system such as those of [8], the average δz is given by the bound of Eq. (20). These two facts imply that the fluctuations of contact numbers are small. Therefore the conclusions of the last section remain valid, and no anomalous modes below the bound ω^* are expected in the relaxed-spring system due to fluctuations of z.

C. Extension to nonharmonic contacts

In the previous sections we considered harmonic interactions. Here we discuss the generalization of our argument to other potentials. Reference [8] explored several other interactions, notably the Hertzian interaction potential describing the compressive energy of two elastic spheres. It corresponds to $\alpha=\frac{5}{2}$ in Eq. (5). Reference [8] observed a plateau in the density of states whose height D_0 scales as $p^{-1/6}$. They also observed a cutoff frequency ω^* varying as $p^{1/2}$. In the Herztian case the quadratic energy of Eq. (7) becomes

$$\delta E=\frac{1}{2}\sum_{\langle ij\rangle}(1-r_{ij})^{1/2}[(\delta\vec{R}_j-\delta\vec{R}_i)\cdot\vec{n}_{ij}]^2. \tag{21}$$

The new factor $(1-r_{ij})^{1/2}$ amounts to a spring constant k_{ij} that depends on compression. The contact force $f_{ij}=\partial\delta E/\partial r_{ij}$ evidently varies as $(1-r_{ij})^{3/2}$. In what follows we neglect the fluctuations that exist between the contacts. This treatment is sufficient to recover the scaling results of [8].

The new factor $(1-r_{ij})^{1/2}$ rescales the energy. To account for this overall effect, we replace $(1-r_{ij})^{1/2}$ by its average $\langle(1-r_{ij})^{1/2}\rangle$. Expressed in terms of contact forces, this factor is proportional to $\langle f_{ij}^{1/3}\rangle$. Replacing f_{ij} by its average, the factor becomes $\langle f_{ij}\rangle^{1/3}$. This average is related to the pressure p, via $p\approx\langle f_{ij}\rangle$. Thus, in this approximation the overall effect is to rescale the energy by a factor $k(p)\sim p^{1/3}$.

$$\delta E=\frac{k(p)}{2}\sum_{\langle ij\rangle}[(\delta\vec{R}_j-\delta\vec{R}_i)\cdot\delta\vec{n}_{ij}]^2. \tag{22}$$

Apart from this prefactor, the energy and the dynamical matrix are identical to the harmonic case treated above. Each normal mode frequency gains a factor $k^{1/2}\sim p^{1/6}$. In the harmonic case the crossover frequency follows $\omega^*\sim\delta z$. In the Hertzian case, it gains the same factor $k^{1/2}$, so that $\omega^*\sim k^{1/2}\delta z$. The bound on the lowest-frequency anomalous modes ω_{AM} still has the form

$$\omega_{AM}{}^2\leqslant\omega^{*2}-A_2p. \tag{23}$$

For a marginally stable system we still have $\omega_{AM}=0$, which leads to an unaltered relationship between ω^* and $p:\omega^*\sim p^{1/2}$. Comparing with our previous estimate of ω^* we find $\delta z\sim p^{1/3}$. Furthermore, the plateau density of states D_0 has the dimension of an inverse frequency and thus gains a factor $p^{-1/6}$. Since the harmonic D_0 had no dependence on p, the Hertzian $D_0(p)$ also should vary as $p^{-1/6}$. The scaling behaviors seen in [8] agree with these expectations. These arguments may be applied to general values of the interaction exponent α.

Additional effects could, in principle, alter the low-frequency modes in the Hertzian case. When harmonic springs are replaced by Hertzian ones, different contacts have different stiffness. This effect should be quantified in order to gain a more detailed understanding of the Hertzian case [37].

VI. CONCLUSIONS

In this paper we computed some of the vibrational properties of weakly connected amorphous systems. Above a frequency scale ω^* such systems do not behave as continuous elastic media. In this regime the vibrations correspond to the anomalous modes that constitute the vibrations of marginally stable, isostatic systems. At frequencies lower than ω^*, the system acts as a continuous solid, with low-frequency acoustic modes. As we showed in Sec. III, these anomalous modes are built from the soft modes that appear in subsystems with free-boundary conditions. At ω^* the anomalous modes are characterized by a length scale l^*. Interestingly l^* does not appear directly in the correlation functions of the static structure, but only in the response functions. l^* can be much larger than the particle size and varies with the coordination number, as $l^*\sim\omega^{*-1}\sim\delta z^{-1}$. Secondly we computed the effect of applied stress on these anomalous modes. In a repulsive system the stress has a strong effect that lowers the frequency of the anomalous modes. Imposing that such modes are stable leads to a generalization of the Maxwell criterion: the coordination must increase nonanalytically with pressure to compensate for the destabilizing effect of compression. Finally, we discussed the “structural buckling” that occurs when anomalous modes collapse. We use this concept to justify the marginal stability that follows hyperquenches in the simulations of repulsive, short-range systems of [8].

The anomalous modes offer a new approach for understanding response and transport in weakly connected mechanical systems. Knowledge of the statistical properties of the anomalous and soft modes defined by Eq. (10) is necessary for predicting the acoustic and thermal transport of, e.g., the simulated system of [8]. These modes also provide an alternative view of relaxation, via buckling of compressed anomalous modes. This buckling picture provides an intriguing contrast to the local cage-escape picture commonly used to describe these relaxations. This approach may be more broadly applicable for explaining the presence of excess modes in some crystals [38] and in glasses [37]. Glasses exhibit a large excess of low-frequency modes like our marginally jammed system; this suggests that they behave like weakly connected mechanical systems at short length scales. This resemblance raises the hope that our approach may explain anomalous phenomena in the transport, response, relaxation, and aging of structural glasses.

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APPENDIX: SPATIAL DISTRIBUTION OF THE SOFT MODES

In our argument we have assumed that when $q \sim L^{d-1}$ contacts were cut along a hyperplane in an isostatic system, there was a vector space of dimension $q' = aq$ which contains only extended modes, and that a does not vanish when $L \to \infty$. A normalized mode $|\delta\mathbf{R}\rangle$ was said to be extended if $\Sigma_i \sin^2(x_i \pi/L)\langle i|\delta\mathbf{R}\rangle^2 > e_0$, where e_0 is a constant, and does not depend on L. Here we show how to chose $e_0 > 0$ so that there is a nonvanishing fraction of extended soft modes. We build the vector space of extended soft modes and furnish a bound to its dimension.

Let us consider the linear mapping $\mathcal{G}$ which assigns us to a displacement field $|\delta\mathbf{R}\rangle$, the displacement field $\langle i|\mathcal{G}\delta\mathbf{R}\rangle = \sin^2(x_i\pi/L)\delta\vec{R}_i$. For any soft mode $|\delta\mathbf{R}_\beta\rangle$ one can consider the positive number $a_\beta \equiv \langle\delta\mathbf{R}_\beta|\mathcal{G}|\delta\mathbf{R}_\beta\rangle \equiv \Sigma_i \sin^2(x_i\pi/L)\delta\vec{R}_{i,\beta}^2$. We build the vector space of extended modes by recurrence: at each step we compute the a_β for the normalized soft modes, and we eliminate the soft mode with the minimum a_β. We then repeat this procedure in the vector space orthogonal to the soft modes eliminated. We stop the procedure

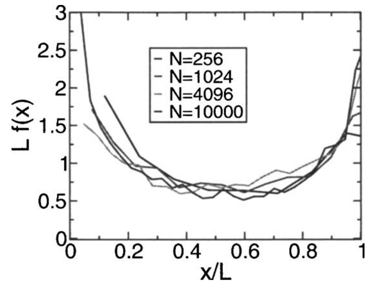

FIG. 8. The overlap function $f(x)$ as defined in Eq. (A1) for different system sizes in three dimensions. Soft modes were created from isostatic configurations as described in Fig. 3.

when $a_\beta > e_0$ for all the soft modes β left. Then all the modes left are extended according to our definition. We just have to show that one can choose $e_0 > 0$ such that when this procedure stops, there are q' modes left, with $q' > aq$, with $a > 0$. In order to show that, we introduce the following overlap function:

$$f(x)dx \equiv q^{-1} \sum_{\beta=1,\ldots,q} \sum_{x_i \in [x,x+dx]} [\delta\vec{R}_{i,\beta}]^2. \tag{A1}$$

The sum is taken on an orthonormal basis of soft modes β and on all the particles whose position has a coordinate $x_i \in [x, x+dx]$. $f(x)$ is the trace of a projection, and is therefore independent of the orthonormal basis considered. $f(x)$ describes the spatial distribution of the amplitude of the soft modes. The $|\delta\mathbf{R}_\beta\rangle$ are normalized and therefore

$$\int_0^L f(x)dx = 1 \tag{A2}$$

We have examined soft modes made from configurations at the jamming transition found numerically in [8]. The overlap function $f(x)$ was then computed for different system sizes L. These are shown in Fig. 8. It appears from Fig. 8 that (i) when $f(x)$ is rescaled with the system size it collapses to a unique curve, and (ii) this curve is bounded below by a constant c ($c \approx 0.6$). Consequently one can bound the trace of $\mathcal{G}$: $\mathrm{tr}\mathcal{G} = \int_0^L q f(x)\sin^2(x) > qc/2$. On the other hand, one has $\mathrm{tr}\mathcal{G} = \Sigma_{\beta=1}^q a_\beta$, where the sum is made on the orthonormal basis we just built in the previous paragraph. This sum can be divided into contributions from the nonextended states and extended states:

$$\sum_{\beta=1}^{q} a_\beta = \sum_{\beta=1}^{q-q'} a_\beta + \sum_{\beta=q-q'+1}^{q} a_\beta. \tag{A3}$$

By construction all the a_β in the first sum are smaller than e_0, while all the a_β in the (first or) second are smaller than 1. Thus $\Sigma_{\beta=1}^q \alpha_\beta < (q-q')e_0 + q'$. Combining, we have $(q-q')e_0 + q' > qc/2$, or

$$q' > q(c/2 - e_0)/(1 - e_0). \tag{A4}$$

Evidently for all L, q' remains a nonvanishing fraction of q for any fixed threshold e_0 such that $e_0 < c/2$, as claimed.

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[22] At the jamming transition the rigid system is a d-dimensional object with only a few holes and the rattlers are typically clusters of one or two particles. This is very different from the rigidity percolation models where bonds are randomly deposited on a lattice. In that case, the percolating rigid cluster is a fractal object with a dimension smaller than d.
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[24] If m_α is the αth lowest eigenvalue of $\mathcal{M}$ and if e_α is an orthonormal basis such that $\langle e_\alpha|\mathcal{M}|e_\alpha\rangle \equiv n_\alpha$ then the variational bound of A. Horn [Am. J. Math. **76**, 620 (1954)] shows that $\Sigma_1^q m_\alpha \geqslant \Sigma_1^q n_\alpha$. Since $qn_q \geqslant \Sigma_1^q n_\alpha$, and since $\Sigma_1^q m_\alpha \geqslant \Sigma_{q/2}^q m_\alpha \geqslant (q/2) m_{q/2}$, we have $qm_p \geqslant (q/2) n_{q/2}$ as claimed.
[25] The balance of force can be satisfied in $\mathcal{S}'$ by imposing external forces on the free boundary. This adds a linear term in the energy expansion that does not affect the normal modes.
[26] In a polydisperse system z_{max} could *a priori* be larger. Nevertheless Eq. (14) is a sum on every contact where the displacement of only one of the two particles appears in each term of the sum. The corresponding particle can be chosen arbitrarily. It is convenient to choose the smallest particle of each contact. Thus when this sum on every contact is written as a sum on every particle to obtain Eq. (15), the constant z_{max} still corresponds to the monodisperse case, as a particle cannot have more contacts than that with particles larger than itself.
[27] In three dimension we have $\mathcal{M}'_{ij} = -(1 - r_{ij}/2r_{ij})[\delta_{\langle ij \rangle}(\vec{m}_{ij} \otimes \vec{m}_{ij} + \vec{k}_{ij} \otimes \vec{k}_{ij}) + \delta_{i,j}\Sigma_{\langle l \rangle}(\vec{m}_{ij} \otimes \vec{m}_{ij} + \vec{k}_{ij} \otimes \vec{k}_{ij})]$, where $(\vec{n}_{ij}, \vec{m}_{ij}, \vec{k}_{ij})$ is an orthonormal basis.
[28] In disordered systems the acoustic modes are not exact acoustic modes, see, e.g., recent simulations in Lennard-Jones systems [6]. Therefore the correction due to the applied pressure on the frequency of the acoustic modes might be larger than what is expected from Eq. (17). Nevertheless, we still expect the effect of applied pressure to be much smaller on the acoustic modes than on the anomalous modes.
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[32] When the particle's radius decreases by an amount ϵ, a certain fraction $e \sim g(1)\epsilon$ of contacts opens, where $g(r)$ denotes the radial distribution function. For harmonic particles we expect $g(1) \sim (\phi - \phi_c)^{-1}$, as we justify at the end of the paragraph. Thus using the fact that $p \sim (\phi - \phi_c)$ for harmonic particles [8], one obtains $e \sim \epsilon/p$. On the other hand, the affine decompression lowers the pressure by an amount $\delta p \sim \delta\phi \sim \epsilon$. Thus the fraction f of contacts that the system can afford to lose while staying stable follows, according to Eq. (20): $f = d(\delta z_{min})/dp\, \delta p \sim p^{-1/2} \delta p \sim \epsilon/p^{1/2} \ll e$. Therefore $f \ll e$ as claimed. Thus if an affine reduction of packing fraction ϵ is imposed, far too many contacts open and the system is unstable. To conclude we justify $g(1) \sim (\phi - \phi_c)^{-1}$. This is related to well-known empirical facts of the force distribution: one has $P(F)dF \sim g(r)dr$. For harmonic particles $dr \sim dF$ and, therefore, $P(F) \sim g(r)$. When rescaled by $\langle F \rangle \sim p, P(F)$ converges to a master curve with $P(F=0) \neq 0$ [5,8]. This implies that $g(1) \sim p^{-1} \sim (\phi - \phi_c)^{-1}$ [31].
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